Computer Engineering Department

Value-based Theoretical Guarantees

Mohammad Hossein Rohban, Ph.D.

Spring 2025

Courtesy: Most of slides are adopted from ML course EE3001 by Jie Wang.

Bellman's Optimality Equation

- Assume a stochastic reward function.

$$\Pr(S_{t+1} = s', R_{t+1} = r | S_t = s, A_t = a), \forall\, s, s' \in \mathcal{S}, r \in \mathcal{R}, a \in \mathcal{A},$$

which is abbreviated by $p(s', r|s, a)$.

$$\begin{aligned}
q_*(s, a) &= \max_\pi \mathbb{E}[G_t | S_t = s, A_t = a] \\
&= \max_\pi \mathbb{E}[R_{t+1} + \gamma G_{t+1} | S_t = s, A_t = a] \\
&= \mathbb{E}[R_{t+1} | S_t = s, A_t = a] + \gamma \max_\pi \mathbb{E}[G_{t+1} | S_t = s, A_t = a].
\end{aligned}$$

Bellman's Optimality Equation (cont.)

$$\mathbb{E}[R_{t+1}|S_t = s, A_t = a] = \sum_r r \sum_{s'} p(s', r|s, a).$$

$$\begin{aligned}
\mathbb{E}[G_{t+1}|S_t = s, A_t = a] &= \sum_{s',a'} p(s', a'|s, a)\mathbb{E}[G_{t+1}|S_{t+1} = s', A_{t+1} = a', S_t = s, A_t = a] \\
&= \sum_{s',a'} p(s'|s, a)p(a'|s', s, a)\mathbb{E}[G_{t+1}|S_{t+1} = s', A_{t+1} = a'] \\
&= \sum_{s',a'} p(s'|s, a)\pi(a'|s')q_\pi(s', a') \\
&= \sum_{s'} p(s'|s, a) \sum_{a'} \pi(a'|s')q_\pi(s', a').
\end{aligned}$$

Bellman's Optimality Equation (cont.)

$$q_*(s,a) = \sum_r r \sum_{s'} p(s', r|s,a) + \gamma \max_\pi \sum_{s'} p(s'|s,a) \sum_{a'} \pi(a'|s') q_\pi(s',a').$$

$$q_*(s,a) = \sum_r r \sum_{s'} p(s', r|s,a) + \gamma \max_\pi \sum_{s'} p(s'|s,a) \max_{a'} q_\pi(s',a').$$

Bellman's Optimality Equation (cont.)

$$\begin{aligned} q_*(s,a) &= \sum_r r \sum_{s'} p(s',r|s,a) + \gamma \sum_{s'} p(s'|s,a) \max_{a'} q_*(s',a') \\ &= \sum_{r,s'} p(s',r|s,a)(r + \gamma \max_{a'} q_*(s',a')). \end{aligned}$$

Questions

- Does there exist q_* functions satisfying the Bellman's Eq.?
- Is this function unique?
- Can value iteration find this function?

Fixed Point

- For an operator T, we call x a fixed point if $Tx = x$.
- q_* is a fixed point of the Bellman's Eq.
- Why?

Fixed Point (cont.)

Theorem 1 (Banach Fixed Point Theorem). *Suppose that X is a nonempty complete metric space and $T : X \to X$ is a contraction mapping on X. Then T has a unique fixed point.*

Definition 1 (Contraction Mapping). [1] Let (X, d) be a metric space. A mapping $T : X \to X$ is called a *contraction mapping* on X if there is a positive real number $\alpha < 1$ such that for any $x, y \in X$

$$d(Tx, Ty) \leq \alpha d(x, y).$$

Existence Proof

- Pick an arbitrary point x_0.
- Construct a sequence: $x_k = Tx_{k-1},\ k = 1, 2, \ldots.$
- Let $C = d(x_1, x_0)$.
- Note that

$$d(x_{k+1}, x_k) \le \alpha d(x_k, x_{k-1}) \le \cdots \le \alpha^k d(x_1, x_0) = \alpha^k C, \forall, k = 1, 2, \ldots.$$

$$d(x_m, x_n) \le \sum^{m-n-1} d(x_{n+i+1}, x_{n+i}).$$

$$d(x_m, x_n) \le \sum_{i=0}^{m-n-1} \alpha^{n+i} C = \alpha^n C \frac{1 - \alpha^{m-n}}{1 - \alpha} \le \alpha^n \frac{C}{1 - \alpha}.$$

Existence Proof

- Thus for any $\epsilon > 0$, if $N \geq \frac{\log \epsilon(1-\alpha)-\log C}{\log \alpha}$ then $d(x_m, x_n) \leq \epsilon.$
- Hence x_n is a Cauchy sequence.
- Therefore, it converges to a point, let's call x.
- Now, we show that x is a fixed point of T.
- Note that:

$$d(Tx, x) \leq d(Tx, x_k) + d(x_k, x) \leq \alpha d(x, x_{k-1}) + d(x_k, x), \forall\, k = 1, 2, \ldots .$$

$$d(Tx, x) = 0,$$

Uniqueness

- Proof by contradiction.
- Let *x′* be another such fixed point.
- Then, $d(x, x') = d(Tx, Tx') \leq \alpha d(x, x'),$
- Which is a contradiction.

Application to the Bellman's Eq.

- Define the operator T as:

$$Tq(s,a) = \sum_{r,s'} p(r, s'|s, a)(r + \gamma \max_{a'} q(s', a')),$$

T in Bellman is contraction

Lemma 1. *For a finite MDP, the mapping T in Eq. (10) is a contraction mapping.*

Proof. We consider the complete metric space $(\mathbb{R}^{|\mathcal{S}|\times|\mathcal{A}|}, d)$, where $d(q_1, q_2) = \|q_1 - q_2\|_\infty$ for any $p, q \in \mathbb{R}^{|\mathcal{S}|\times|\mathcal{A}|}$. Then,

$$\begin{aligned}
\|Tq_1 - Tq_2\|_\infty =& \max_{s,a} |Tq_1(s,a) - Tq_2(s,a)| \\
=& \gamma \max_{s,a} \sum_{r,s'} p(r, s'|s,a) |\max_{a'} q_1(s',a') - \max_{a'} q_2(s',a')| \\
\leq& \gamma \max_{s,a} \sum_{s'} p(s'|s,a) \max_{a'} |q_1(s',a') - q_2(s',a')| \\
\leq& \gamma \max_{s,a} \max_{s'} \max_{a'} |q_1(s',a') - q_2(s',a')| \\
=& \gamma \max_{s',a'} |q_1(s',a') - q_2(s',a')| \\
=& \gamma \|q_1 - q_2\|_\infty,
\end{aligned}$$

Why value iteration converges to the fixed point?

- Let’s discuss!