Computer Engineering Department

# Value-based Theoretical Guarantees

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Courtesy: Most of slides are adopted from ML course EE3001 by Jie Wang.

# Bellman's Optimality Equation

- Assume a stochastic reward function.

$$\Pr(S_{t+1} = s', R_{t+1} = r | S_t = s, A_t = a), \forall\, s, s' \in \mathcal{S}, r \in \mathcal{R}, a \in \mathcal{A},$$

which is abbreviated by $p(s', r|s, a)$.

$$\begin{aligned} q_*(s,a) &= \max_\pi \mathbb{E}[G_t | S_t = s, A_t = a] \\ &= \max_\pi \mathbb{E}[R_{t+1} + \gamma G_{t+1} | S_t = s, A_t = a] \\ &= \mathbb{E}[R_{t+1} | S_t = s, A_t = a] + \gamma \max_\pi \mathbb{E}[G_{t+1} | S_t = s, A_t = a]. \end{aligned}$$

# Bellman's Optimality Equation (cont.)

$$\mathbb{E}[R_{t+1}|S_t = s, A_t = a] = \sum_r r \sum_{s'} p(s', r|s, a).$$

$$\begin{aligned}
\mathbb{E}[G_{t+1}|S_t = s, A_t = a] &= \sum_{s',a'} p(s', a'|s, a)\mathbb{E}[G_{t+1}|S_{t+1} = s', A_{t+1} = a', S_t = s, A_t = a] \\
&= \sum_{s',a'} p(s'|s, a)p(a'|s', s, a)\mathbb{E}[G_{t+1}|S_{t+1} = s', A_{t+1} = a'] \\
&= \sum_{s',a'} p(s'|s, a)\pi(a'|s')q_\pi(s', a') \\
&= \sum_{s'} p(s'|s, a) \sum_{a'} \pi(a'|s')q_\pi(s', a').
\end{aligned}$$

# Bellman’s Optimality Equation (cont.)

$$q_*(s,a) = \sum_r r \sum_{s'} p(s', r|s, a) + \gamma \max_\pi \sum_{s'} p(s'|s, a) \sum_{a'} \pi(a'|s') q_\pi(s', a').$$

$$q_*(s,a) = \sum_r r \sum_{s'} p(s', r|s, a) + \gamma \max_\pi \sum_{s'} p(s'|s, a) \max_{a'} q_\pi(s', a').$$

# Bellman's Optimality Equation (cont.)

$$
\begin{aligned}
q_*(s,a) &= \sum_r r \sum_{s'} p(s', r|s,a) + \gamma \sum_{s'} p(s'|s,a) \max_{a'} q_*(s',a') \\
&= \sum_{r,s'} p(s', r|s,a)(r + \gamma \max_{a'} q_*(s',a')).
\end{aligned}
$$

# Questions

- Does there exist $q_*$ functions satisfying the Bellman's Eq.?
- Is this function unique?
- Can value iteration find this function?

# Fixed Point

$T(\underline{x})$

- For an operator $T$, we call $x$ a fixed point if $Tx = x$.
- $q_*$ is a fixed point of the Bellman's Eq.
- Why?

$$x = \begin{bmatrix} q^*(s_1, a_1) \\ q^*(s_2, a_1) \\ \vdots \\ q^*(s_n, a_m) \end{bmatrix}$$

$$\underline{x} = T\underline{x}$$

$$T = \begin{bmatrix} f_{1,1} \\ \vdots \\ f \end{bmatrix} \qquad f_{i,j}(\underline{x})$$

$$= q^*(s_i, a_j)$$

$$= \sum_{s', r} p(s', r \mid s_i, a_j)\left[r + \gamma \overbrace{\max_{a'} q^*(s', a')}^{\underline{x}}\right]$$

# Fixed Point (cont.)

**Theorem 1 (Banach Fixed Point Theorem).** *Suppose that $X$ is a nonempty complete metric space and $T : X \to X$ is a contraction mapping on $X$. Then $T$ has a unique fixed point.*

**Definition 1 (Contraction Mapping).** [1] Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is called a *contraction mapping* on $X$ if there is a positive real number $\alpha < 1$ such that for any $x, y \in X$

$$d(Tx, Ty) \leq \alpha d(x, y).$$

# Existence Proof

Cauchy Seq. $x_n$ if $\forall \epsilon$ $\exists N$ : $m, n \geq N \Rightarrow$
$d(x_m, x_n) \leq \epsilon$

- Pick an arbitrary point $x_0$.
- Construct a sequence: $x_k = Tx_{k-1},\ k = 1, 2, \ldots.$
- Let $C = d(x_1, x_0)$.
- Note that

$$0 \leq d(\underbrace{x_{k+1}}_{Tx_k}, \underbrace{x_k}_{Tx_{k-1}}) \leq \alpha d(\overbrace{x_k}^{Tx_{k-1}}, \overbrace{x_{k-1}}^{Tx_{k-2}}) \leq \cdots \leq \alpha^k d(x_1, x_0) = \alpha^k C,\ \forall, k = 1, 2, \ldots.$$

$$d(x_m, x_n) \leq \sum^{m-n-1} d(\overbrace{x_{n+i+1}}^{k+1}, \overbrace{x_{n+i}}^{k}).$$

$$d(x_m, x_n) \leq \sum_{i=0}^{m-n-1} \alpha^{n+i} C = \alpha^n C \frac{1 - \alpha^{m-n}}{1 - \alpha} \leq \alpha^n \frac{C}{1 - \alpha}. = \epsilon$$

# Existence Proof

- Thus for any $\epsilon > 0$, if $N \geq \frac{\log \epsilon(1-\alpha) - \log C}{\log \alpha}$ then $d(x_m, x_n) \leq \epsilon$.
- Hence $x_n$ is a Cauchy sequence.
- Therefore, it converges to a point, let's call *x*.
- Now, we show that *x* is a fixed point of *T*.
- Note that:


$$d(Tx, x) \leq d(Tx, \underbrace{x_k}_{Tx_{k-1}}) + d(x_k, x) \leq \alpha d(x, x_{k-1}) + d(x_k, x), \; \forall\, k = 1, 2, \ldots.$$

$$d(Tx, x) = 0,$$

# Uniqueness

- Proof by contradiction.
- Let *x′* be another such fixed point.
- Then, $d(x, x') = d(Tx, Tx') \leq \alpha d(x, x'),$
- Which is a contradiction.

# Application to the Bellman's Eq.

- Define the operator *T* as:

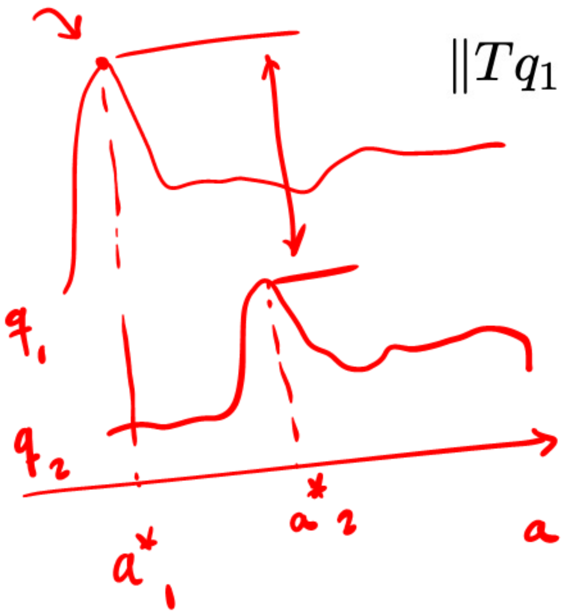
$$Tq(s,a) = \sum_{r,s'} p(r, s'|s, a)(r + \gamma \max_{a'} q(s', a')),$$

# T in Bellman is contraction

**Lemma 1.** *For a finite MDP, the mapping $T$ in Eq. (10) is a contraction mapping.*

*Proof.* We consider the complete metric space $(\mathbb{R}^{|\mathcal{S}|\times|\mathcal{A}|}, d)$, where $d(q_1, q_2) = \|q_1 - q_2\|_\infty$ for any $p, q \in \mathbb{R}^{|\mathcal{S}|\times|\mathcal{A}|}$. Then,

$\|\underline{x}\|_\infty = \max(|x_1|, \dots, |x_d|)$




$$
\begin{aligned}
\|Tq_1 - Tq_2\|_\infty &= \max_{s,a} |Tq_1(s,a) - Tq_2(s,a)| \\
&= \gamma \max_{s,a} \sum_{r,s'} p(r, s'|s,a) |\max_{a'} q_1(s',a') - \max_{a'} q_2(s',a')| \\
&\leq \gamma \max_{s,a} \sum_{s'} p(s'|s,a) \max_{a'} |q_1(s',a') - q_2(s',a')| \\
&\leq \gamma \max_{s,a} \max_{s'} \max_{a'} |q_1(s',a') - q_2(s',a')| \\
&= \gamma \max_{s',a'} |q_1(s',a') - q_2(s',a')| \\
&= \gamma \|q_1 - q_2\|_\infty,
\end{aligned}
$$

$\left|\sum_{r,s'} p(r,s'|s,a)\right.$
$\left[r + \gamma \max_{a'} q_1(s',a')\right.$
$- r + \gamma \max_{a'} q_2(s',a')$

# Why value iteration converges to the fixed point?

- Let’s discuss!

# Policy Improvement Improves!

- If we set the new policy to maximize q(s, a) over a, the new policy leads to higher v(s) values for all states s.
- Let's discuss!

# Policy Iteration Converges

**Theorem.** Policy iteration is guaranteed to converge and at convergence, the current policy and its value function are the optimal policy and the optimal value function!

Proof sketch:

(1) *Guarantee to converge*: In every step the policy improves. This means that a given policy can be encountered at most once. This means that after we have iterated as many times as there are different policies, i.e., (number actions)$^{\text{(number states)}}$, we must be done and hence have converged.

(2) *Optimal at convergence*: by definition of convergence, at convergence $\pi_{k+1}(s) = \pi_k(s)$ for all states s. This means $\forall s \;\; V^{\pi_k}(s) = \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_i^{\pi_k}(s') \right]$

Hence $V^{\pi_k}$ satisfies the Bellman equation, which means $V^{\pi_k}$ is equal to the optimal value function V*.